Crystal Bases and Affine Deligne-Lusztig Varieties

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Abstract

The affine Deligne-Lusztig variety was introduced by Rapoport in [8], which plays an important role in understanding Shimura varieties. There are two combinatorial ways of parameterizing the $J_b(F)$-orbits of the irreducible components of affine Deligne-Lusztig varieties for GL_n and superbasic b. One way is to use the extended semi-modules introduced by Viehmann. The other way is to use the crystal bases introduced by Kashiwara and Lusztig. Based on [9], we explain an explicit correspondence between them using the crystal structure.

1 Introduction

Let F be a non-archimedean local field with finite field $\mathbb{F}_q$ of prime characteristic p, and let L be the completion of the maximal unramified extension of F. Let σ denote the Frobenius automorphism of L/F. Further, we write $\mathcal{O}$, $\mathfrak{p}$ for the valuation ring and the maximal ideal of L. Finally, we denote by ϖ a uniformizer of F (and L) and by v_L the valuation of L such that $v_L(\varpi) = 1$.

Let G be a split connected reductive group over F and let T be a split maximal torus of it. Let B be a Borel subgroup of G containing T. For a cocharacter $\mu \in X_*(T)$, let ϖ^μ be the image of $\varpi \in \mathbb{G}_m(F)$ under the homomorphism $\mu\colon \mathbb{G}_m \to T$.

Set $K = G(\mathcal{O})$. We fix a dominant cocharacter $\mu \in X_*(T)_+$ and $b \in G(L)$. Then the affine Deligne-Lusztig variety $X_\mu(b)$ is the locally closed reduced $\overline{\mathbb{F}}_q$-subscheme of the affine Grassmannian $\mathcal{G}r$ defined as

$$X_\mu(b)(\overline{\mathbb{F}}_q) = \{xK \in G(L)/K \mid x^{-1}b\sigma(x) \in K\varpi^\mu K\} \subset \mathcal{G}r(\overline{\mathbb{F}}_q).$$

Left multiplication by $g^{-1} \in G(L)$ induces an isomorphism between $X_\mu(b)$ and $X_\mu(g^{-1}b\sigma(g))$. Thus the isomorphism class of the affine Deligne-Lusztig variety only depends on the σ-conjugacy class of b.

The affine Deligne-Lusztig variety $X_\mu(b)$ carries a natural action (by left multiplication) by the group

$$J_b(F) = \{g \in G(L) \mid g^{-1}b\sigma(g) = b\}.$$

For $\mu_\bullet = (\mu_1, \ldots, \mu_d) \in X_*(T)_+^d$ and $b_\bullet = (1, \ldots, 1, b) \in G^d(L)$ with $b \in G(L)$, we can similarly define $X_{\mu_\bullet}(b_\bullet) \subset \mathcal{G}r^d$ and $J_{b_\bullet}(F)$ using $\sigma_\bullet$ given by

$$(g_1, g_2, \ldots, g_d) \mapsto (g_2, \ldots, g_d, \sigma(g_1)).$$

The geometric properties of affine Deligne-Lusztig varieties have been studied by many people. One of the most interesting results is an explicit description of the set $J_b(F)\backslash \operatorname{Irr} X_\mu(b)$ of $J_b(F)$-orbits of $\operatorname{Irr} X_\mu(b)$, where $\operatorname{Irr} X_\mu(b)$ denotes the set of irreducible components of $X_\mu(b)$ (it is known that $X_\mu(b)$ is equi-dimensional).

Let $\widehat{G}$ be the Langlands dual of G defined over $\overline{\mathbb{Q}}_l$ with $l \neq p$. Denote V_μ the irreducible $\widehat{G}$-module of highest weight μ. The crystal basis $\mathbb{B}_\mu$ of V_μ was first constructed by Kashiwara and Lusztig (cf. [4]). In $X_*(T)$, there is a distinguished element λ_b determined by b. It is the "best integral approximation" of the Newton vector of b, but we omit the precise definition. For this, see [2, §2.1] or [2, Example 2.3]. In [7], Nie proved that there exists a natural bijection

$$J_b(F)\backslash \operatorname{Irr} X_\mu(b) \cong \mathbb{B}_\mu(\lambda_b).$$

In particular, $|J_b(F)\backslash \operatorname{Irr} X_\mu(b)| = \dim V_\mu(\lambda_b)$. The proof is reduced to the case where $G = \mathrm{GL}_n$ and b is superbasic. So this case is particularly important. This theorem is first conjectured by Miaofen Chen and Xinwen Zhu. Before the work by Nie, Xiao-Zhu [11] proved the conjecture under the assumption that b is unramified, and Hamacher-Viehmann [2] proved the minuscule case. The last equality is also proved by Rong Zhou and Yihang Zhu in [12]. See [12, §1.2] for the history.

On the other hand, in the case where $G = \mathrm{GL}_n$ and b is superbasic, Viehmann [10] defined a stratification of $X_\mu(b)$ using extended semi-modules. For $\mu \in X_*(T)_+$ and superbasic $b \in \mathrm{GL}_n(L)$, let $\mathbb{A}_{\mu,b}^{\mathrm{top}}$ be the set of top extended semi-modules (cf. §2.2), that is, the extended semi-modules whose corresponding strata are top-dimensional. Then $J_b(F)\backslash \operatorname{Irr} X_\mu(b)$ is also parametrized by $\mathbb{A}_{\mu,b}^{\mathrm{top}}$.

In [7, Remark 0.10], Nie pointed out that it would be interesting to give an explicit correspondence between $\mathbb{A}_{\mu,b}^{\mathrm{top}}$ and $\mathbb{B}_\mu(\lambda_b)$. The purpose of this article is to study this question (for the split case). More precisely, we will propose a way of constructing (the unique lifts of) all the top extended semi-modules from crystal elements, which was unclear before this work.

From now and until the end of this article, we set $G = \mathrm{GL}_n$. Let T be the torus of diagonal matrices, and we choose the subgroup of upper triangular matrices B as Borel subgroup. Let us define the Iwahori subgroup $I \subset K$ as the inverse image of the *lower* triangular matrices under the projection $K \to G(\overline{\mathbb{F}}_q)$, $\varpi \mapsto 0$.

We assume b to be superbasic, i.e., its Newton vector $\nu_b \in X_*(T)_\mathbb{Q} \cong \mathbb{Q}^n$ is of the form $\nu_b = (\frac{m}{n}, \ldots, \frac{m}{n})$ with $(m, n) = 1$. Moreover, we choose b to be η^m, where $\eta = \begin{pmatrix} 0 & \varpi \\ 1_{n-1} & 0 \end{pmatrix}$. We often regard η (and hence b) as an element of the Iwahori-Weyl group $\widetilde{W}$. For superbasic b, the condition that $X_\mu(b)$ (resp. $X_{\mu_\bullet}(b_\bullet)$) is non-empty is equivalent to $v_L(\det(\varpi^\mu)) = v_L(\det(b))$ (resp. $v_L(\det(\varpi^{\mu_1+\cdots+\mu_d})) = v_L(\det(b))$) (cf. [3, Theorem 3.1]). In this article, we assume this.

Since $X_\mu(b) = X_{\mu+c}(\varpi^c b)$ for any central cocharacter c, we may assume that $\mu(1) \geq \cdots \geq \mu(n-1) \geq \mu(n) = 0$, where $\mu(i)$ denotes the i-th entry of μ.

To state the main result, we introduce $\mathcal{A}^{\mathrm{top}}_{\mu_\bullet,b_\bullet}$ and $\mathbb{A}^{\mathrm{top}}_{\mu_\bullet,b_\bullet}$. See §4.1 for details. For minuscule $\mu_\bullet \in X_*(T)^d_+$ and $b_\bullet = (1,\ldots,1,b) \in G^d(L)$, we define

$$\mathcal{A}^{\mathrm{top}}_{\mu_\bullet,b_\bullet} \coloneqq \{\lambda_\bullet \in X_*(T)^d \mid \dim X^{\lambda_\bullet}_{\mu_\bullet}(b_\bullet) = \dim X_{\mu_\bullet}(b_\bullet)\}.$$

Here $X^{\lambda_\bullet}_{\mu_\bullet}(b_\bullet)$ denotes $X_{\mu_\bullet}(b_\bullet) \cap It^{\lambda_\bullet}K/K$. For $\lambda_\bullet, \lambda'_\bullet \in \mathcal{A}^{\mathrm{top}}_{\mu_\bullet,b_\bullet}$, we write $\lambda_\bullet \sim \lambda'_\bullet$ if $\lambda_\bullet = \eta^k\lambda'_\bullet = (\eta^k\lambda'_1,\ldots,\eta^k\lambda'_d)$ for some $k \in \mathbb{Z}$. Let $\mathbb{A}^{\mathrm{top}}_{\mu_\bullet,b_\bullet}$ denote the set of equivalence classes with respect to $\sim$, and let $[\lambda_\bullet] \in \mathbb{A}^{\mathrm{top}}_{\mu_\bullet,b_\bullet}$ denote the equivalence class represented by $\lambda_\bullet \in \mathcal{A}^{\mathrm{top}}_{\mu_\bullet,b_\bullet}$. Then $J_{b_\bullet}(F)\backslash \operatorname{Irr} X_{\mu_\bullet}(b_\bullet)$ is parametrized by $\mathbb{A}^{\mathrm{top}}_{\mu_\bullet,b_\bullet}$.

For $\mu \in X_*(T)_+$, let $\mu_\bullet \in X_*(T)^d_+$ be a certain minuscule dominant cocharacter with $\mu = \mu_1 + \mu_2 + \cdots + \mu_n$, see §4.2. Note that $\{\mu_1, \mu_2, \ldots, \mu_n\}$ itself is uniquely determined by μ. Let $\mathrm{pr}\colon \mathcal{G}r^d \to \mathcal{G}r$ be the projection to the first factor. This induces $\mathrm{pr}\colon \mathbb{A}^{\mathrm{top}}_{\mu_\bullet,b_\bullet} \to \sqcup_{\mu' \leq \mu} \mathbb{A}^{\mathrm{top}}_{\mu',b}$. Then our main result is the following:

Theorem A (Theorem 4.4)**.** For $\mathbf{b} \in \mathbb{B}_\mu(\lambda_b)$, using the crystal structure of $\mathbb{B}_\mu$, we can construct $\lambda^1_\bullet(\mathbf{b}), \lambda^2_\bullet(\mathbf{b}), \ldots, \lambda^n_\bullet(\mathbf{b}) \in \mathcal{A}^{\mathrm{top}}_{\mu_\bullet,b_\bullet}$ such that $\lambda^i_\bullet(\mathbf{b}) = \eta^{i-1}\lambda^1_\bullet(\mathbf{b})$ and $[\lambda^1_\bullet(\mathbf{b})]$ is the unique equivalence class in $\mathbb{A}^{\mathrm{top}}_{\mu_\bullet,b_\bullet}$ whose image $\mathrm{pr}([\lambda^1_\bullet(\mathbf{b})])$ belongs to $\mathbb{A}^{\mathrm{top}}_{\mu,b}$ and maps to $\mathbf{b}$ under the bijection $J_b(F)\backslash \operatorname{Irr} X_\mu(b) \cong \mathbb{B}_\mu(\lambda_b)$ by Nie.

A crystal is a finite set with a weight map wt and Kashiwara operators $\tilde{e}_\alpha$ and $\tilde{f}_\alpha$ satisfying certain conditions, see §3. For more details on the construction of $\lambda^1_\bullet(\mathbf{b}), \lambda^2_\bullet(\mathbf{b}), \ldots, \lambda^n_\bullet(\mathbf{b})$, see §4.2. The merit of constructing $[\lambda^1_\bullet(\mathbf{b})]$ instead of constructing $\mathrm{pr}([\lambda^1_\bullet(\mathbf{b})])$ directly is that the $J_b(F)$-orbit in $X_\mu(b)$ corresponding $[\lambda^1_\bullet(\mathbf{b})]$ is much more explicit. It is just $J_b(F)\,\mathrm{pr}(\overline{X^{\lambda^1_\bullet(\mathbf{b})}_{\mu_\bullet}(b_\bullet)})$.

2 Notations

Keep the notations and assumptions in §1.

2.1 Basic Notations

Let $\Phi = \Phi(G,T)$ denote the set of roots of T in G. We denote by Φ_+ (resp. Φ_-) the set of positive (resp. negative) roots distinguished by B. Let χ_{ij} be the character $T \to \mathbb{G}_m$ defined by $\mathrm{diag}(t_1, t_2, \ldots, t_n) \mapsto t_i {t_j}^{-1}$. Using this notation, we have $\Phi = \{\chi_{i,j} \mid i \neq j\}$, $\Phi_+ = \{\chi_{i,j} \mid i < j\}$ and $\Phi_- = \{\chi_{i,j} \mid i > j\}$. Let $\Delta = \{\chi_{i,i+1} \mid 1 \leq i < n\}$ be the set of simple roots and $\Delta^\vee$ be the corresponding set of simple coroots. We let

$$X_*(T)_+ = \{\mu \in X_*(T) | \langle \alpha, \mu \rangle \geq 0 \text{ for all } \alpha \in \Phi_+\}$$

denote the set of dominant cocharacters. Through the isomorphism $X_*(T) \cong \mathbb{Z}^n$, $X_*(T)_+$ can be identified with the set $\{(m_1, \cdots, m_n) \in \mathbb{Z}^n | m_1 \geq \cdots \geq m_n\}$. For $\lambda, \mu \in X_*(T)$, we write $\lambda \leq \mu$ if $\mu - \lambda$ is a linear combination of simple coroots with non-negative coefficients.

Let W_0 denote the finite Weyl group of G, i.e., the symmetric group of degree n. For $1 \leq i \leq n-1$, let s_i be the adjacent transposition changing i to $i+1$. Then $(W_0, \{s_1, \ldots, s_{n-1}\})$ is a Coxeter system, and we denote by ℓ the associated length function. Let $\leq$ denote the Bruhat order on (W_0, S). For $w \in W_0$, we denote by $\mathrm{supp}(w)$ the set of integers $1 \leq i \leq n-1$ such that the simple reflection s_i appears in some/any reduced expression of w. We say $w \in W_0$ is a Coxeter element (resp. partial Coxeter element) if it is a product of simple reflections, and each simple reflection appears exactly once (resp. at most once). Let $\widetilde{W}$ be the Iwahori-Weyl group of G. Then $\widetilde{W}$ is isomorphic to

$$X_*(T) \rtimes W_0 = \{\varpi^\lambda w \mid \lambda \in X_*(T), w \in W_0\},$$

and acts on $X_*(T)$. The action of $\varpi^\lambda w \in \widetilde{W}$ is given by $v \mapsto w(v) + \lambda$.

2.2 Extended Semi-Modules

Here we briefly summarize the definition of extended semi-modules in a combinatorial way, although we do not need it in this article. See [10] for the precise definition. Recall that $b \in G(L)$ is a superbasic element with slope $\frac{m}{n}$.

Definition 2.1. A *semi-module* for m, n is a subset $A \subset \mathbb{Z}$ that is bounded below and satisfies $m + A \subset A$ and $n + A \subset A$. Set $\bar{A} = A \setminus (n + A)$. The semi-module A is called normalized if $\sum_{a \in \bar{A}} a = \frac{n(n-1)}{2}$. An *extended semi-module* (A, φ) for μ is a normalized semi-module A for m, n together with a function $\varphi \colon \mathbb{Z} \to \mathbb{N} \cup \{-\infty\}$ satisfying certain conditions.

Set $X_\mu(b)^0 = \{xK \in X_\mu(b) \mid v_L(\det(x)) = 0\}$. For an extended semi-module (A, φ), we can define a locally closed subset $S_{A,\varphi} \subset X_\mu(b)^0$. They define a decomposition of $X_\mu(b)^0$ into finitely many disjoint locally closed subschemes. Moreover, $S_{A,\varphi} \subset X_\mu(b)^0$ is irreducible. So $J_b(F) \backslash \operatorname{Irr} X_\mu(b)$ is parametrized by $\mathbb{A}_{\mu,b}^{\mathrm{top}} := \{(A, \varphi) \mid$

$\dim S_{A,\varphi} = \dim X_\mu(b)\}$. In [10], extended semi-modules were used to prove the dimension formula (for $X_\mu(b) \neq \emptyset$)

$$\dim X_\mu(b) = \langle \rho, \mu - \nu_b \rangle - \frac{1}{2} \operatorname{def}(b).$$

Here ρ denotes half the sum of positive roots, and $\operatorname{def}(b)$ denotes the defect of b.

Let us also make a few remarks on $\mathcal{A}^{\mathrm{top}}_{\mu_\bullet, b_\bullet}$ introduced in §1. Set $R_{\mu_\bullet, b_\bullet}(\lambda_\bullet) = \{(l, \chi_{i,j}) \mid 1 \leq l \leq d, \langle \chi_{i,j}, \lambda_l^\natural \rangle = -1, (\lambda_l)_{\chi_{i,j}} \geq 1\}$. See §4.1 for the notation. By [7, Proposition 2.9], $X^{\lambda_\bullet}_{\mu_\bullet}(b_\bullet) \neq \emptyset$ if and only if $\lambda^\natural_\bullet$ is conjugate to $\mu_\bullet$. Moreover, in this case,

$$\dim X^{\lambda_\bullet}_{\mu_\bullet}(b_\bullet) = |R_{\mu_\bullet, b_\bullet}(\lambda_\bullet)|.$$

Combining this with the dimension formula for $X_\mu(b)$, we have

$$\mathcal{A}^{\mathrm{top}}_{\mu_\bullet, b_\bullet} = \{\lambda_\bullet \in X_*(T)^d \mid \lambda^\natural_\bullet \in W_0 \mu_\bullet, |R_{\mu_\bullet, b_\bullet}(\lambda_\bullet)| = \langle \rho, \mu - \nu_b \rangle - \frac{1}{2} \operatorname{def}(b)\}.$$

Thus we can actually define $\mathcal{A}^{\mathrm{top}}_{\mu_\bullet, b_\bullet}$ without using affine Deligne-Lusztig varieties.

If $d = 1$, $\mathcal{A}^{\mathrm{top}}_{\mu_\bullet, b_\bullet}$ can be canonically identified with $\mathcal{A}^{\mathrm{top}}_{\mu, b}$. This follows from the fact that if μ is minuscule, then all extended semi-modules for μ are cyclic ([10, COROLLARY 3.7]).

3 Crystal Bases

Keep the notations and assumptions above.

3.1 Crystals and Young Tableaux

In this subsection, we first recall the definition of $\widehat{G}$-crystals from [11, Definition 3.3.1]. After that, we give a realization of crystals by Young tableaux. This allows us to treat them in a combinatorial way.

Definition 3.1. A (normal) $\widehat{G}$-*crystal* is a finite set $\mathbb{B}$, equipped with a weight map $\mathrm{wt}\colon \mathbb{B} \to X_*(T)$, and operators $\tilde{e}_\alpha, \tilde{f}_\alpha \colon \mathbb{B} \to \mathbb{B} \cup \{0\}$ for each $\alpha \in \Delta$, such that

(i) for every $\mathbf{b} \in \mathbb{B}$, either $\tilde{e}_\alpha \mathbf{b} = 0$ or $\mathrm{wt}(\tilde{e}_\alpha \mathbf{b}) = \mathrm{wt}(\mathbf{b}) + \alpha^\vee$, and either $\tilde{f}_\alpha \mathbf{b} = 0$ or $\mathrm{wt}(\tilde{f}_\alpha \mathbf{b}) = \mathrm{wt}(\mathbf{b}) - \alpha^\vee$,

(ii) for all $\mathbf{b}, \mathbf{b}' \in \mathbb{B}$ one has $\mathbf{b}' = \tilde{e}_\alpha \mathbf{b}$ if and only if $\mathbf{b} = \tilde{f}_\alpha \mathbf{b}'$, and

(iii) if $\varepsilon_\alpha, \phi_\alpha \colon \mathbb{B} \to \mathbb{Z},\ \alpha \in \Delta$ are the maps defined by

$$\varepsilon_\alpha(\mathbf{b}) = \max\{k \mid \tilde{e}_\alpha^k \mathbf{b} \neq 0\} \quad \text{and} \quad \phi_\alpha(\mathbf{b}) = \max\{k \mid \tilde{f}_\alpha^k \mathbf{b} \neq 0\},$$

then we require $\phi_\alpha(\mathbf{b}) - \varepsilon_\alpha(\mathbf{b}) = \langle \alpha, \mathrm{wt}(\mathbf{b}) \rangle$.

For $\lambda \in X_*(T)$, we denote by $\mathbb{B}(\lambda)$ the set of elements with weight λ for $\widehat{G}$, called the *weight space* with weight λ for $\widehat{G}$. Let $\mathbb{B}_1$ and $\mathbb{B}_2$ be the two $\widehat{G}$-crystals. A morphism $\mathbb{B}_1 \to \mathbb{B}_2$ is a map of underlying sets compatible with $\mathrm{wt}, \tilde{e}_\alpha$ and $\tilde{f}_\alpha$.

In the sequel, we write $\tilde{e}_i$ and $\tilde{f}_i$ (resp. ε_i and ϕ_i) instead of $\tilde{e}_{\chi_{i,i+1}}$ and $\tilde{f}_{\chi_{i,i+1}}$ (resp. $\varepsilon_{\chi_{i,i+1}}$ and $\phi_{\chi_{i,i+1}}$) for simplicity.

Let $\mathbb{B}_\mu$ be the crystal basis of the irreducible $\widehat{G}$-module of highest weight $\mu \in X_*(T)_+$. Then $\mathbb{B}_\mu$ is a crystal. We call $\mathbb{B}_\mu$ a *highest weight crystal* of highest weight μ (cf. [11, Definition 3.3.1 (3)]). There exists a unique element $\mathbf{b}_\mu \in \mathbb{B}_\mu$ satisfying $\tilde{e}_\alpha \mathbf{b}_\mu = 0$ for all α, $\mathrm{wt}(\mathbf{b}_\mu) = \mu$, and $\mathbb{B}_\mu$ is generated from $\mathbf{b}_\mu$ by operators $\tilde{f}_\alpha$.

We can also define the tensor product of $\widehat{G}$-crystals (cf. [11, Definition 3.3.1(5)]). Taking tensor product of $\widehat{G}$-crystal is associative, making the category of $\widehat{G}$-crystals a monoidal category. Using this fact, we can endow a $\widehat{G}$-crystal structure on the set of semistandard Young tableaux $\mathcal{B}(Y)$ (cf. [4, chapter 7]). For a semistandard tableau $\mathbf{b} \in \mathcal{B}(Y)$, let k_i denote the number of i's appearing in $\mathbf{b}$. Then the weight map wt on this $\widehat{G}$-crystal structure is given by $\mathrm{wt}(\mathbf{b}) = (k_1, \ldots, k_n)$. For an explicit description of the actions of $\tilde{e}_i$ and $\tilde{f}_i$ on $\mathcal{B}(Y)$, see [6, Theorem 3.4.2]. Finally, the following well-known theorem gives a realization of $\mathbb{B}_\mu$.

Theorem 3.2. Let $\mu = (\mu(1), \ldots, \mu(n)) \in X_*(T)_+ \setminus \{0\}$ with $\mu(n) \geq 0$. Let Y be the Young diagram having $\mu(i)$ boxes in the ith row. Then $\mathbb{B}_\mu \cong \mathcal{B}(Y)$.

In the sequel, we identify $\mathbb{B}_\mu$ and $\mathcal{B}(Y)$ by this isomorphism.

Finally, we recall the Weyl group action on crystals. Let $\mathbb{B}$ be a $\widehat{G}$-crystal. For any $1 \leq i \leq n-1$ and $\mathbf{b} \in \mathbb{B}$, we set

$$s_i \mathbf{b} = \begin{cases} \tilde{f}_i^{\langle \chi_{i,i+1}, \mathrm{wt}(\mathbf{b}) \rangle} \mathbf{b} & \text{if } \langle \chi_{i,i+1}, \mathrm{wt}(\mathbf{b}) \rangle \geq 0 \\ \tilde{e}_i^{-\langle \chi_{i,i+1}, \mathrm{wt}(\mathbf{b}) \rangle} \mathbf{b} & \text{if } \langle \chi_{i,i+1}, \mathrm{wt}(\mathbf{b}) \rangle \leq 0. \end{cases}$$

Then we have the obvious relation

$$\mathrm{wt}(s_i \mathbf{b}) = s_i(\mathrm{wt}(\mathbf{b})).$$

By [5, Theorem 7.2.2], this extends to the action of the Weyl group W_0 on $\mathbb{B}$, which is compatible with the action on $X_*(T)$. One can easily verify the following lemma.

Lemma 3.3. Let $w, w' \in W_0$ and $\mathbf{b} \in \mathbb{B}$. If $w(\mathrm{wt}(\mathbf{b})) = w'(\mathrm{wt}(\mathbf{b}))$, then $w\mathbf{b} = w'\mathbf{b}$.

Let $\mathbf{b} \in \mathbb{B}(\lambda)$. If λ' is a conjugate of λ, i.e., there exists $w \in W_0$ such that $\lambda' = w\lambda$, then we call $w\mathbf{b}$ the conjugate of $\mathbf{b}$ with weight λ'. By Lemma 3.3, this does not depend on the choice of w.

3.2 The Minuscule Case

If $\mu \in X_*(T)_+$ is minuscule, then $\mathrm{wt}\colon \mathbb{B}_\mu \to X_*(T)$ gives an identification between $\mathbb{B}_\mu$ and the set of cocharacters which are conjugate to μ. Suppose $\mu_\bullet = (\mu_1, \ldots, \mu_d) \in X_*(T)_+^d$ is minuscule. We can also identify $\mathbb{B}_{\mu_\bullet}^{\widehat{G}^d} := \mathbb{B}_{\mu_1} \times \cdots \times \mathbb{B}_{\mu_d}$ with the set of cocharacters in $X_*(T)^d$ which are conjugate to $\mu_\bullet$. Under this identification, set

$$\mathbb{B}_{\mu_\bullet}^{\widehat{G}^d}(\lambda) = \{(\mu'_1, \ldots, \mu'_d) \in \mathbb{B}_{\mu_\bullet}^{\widehat{G}^d} \mid \mu'_1 + \cdots + \mu'_d = \lambda\}$$

for any $\lambda \in X_*(T)$.

We write $\mathbb{B}_{\mu_\bullet}^{\widehat{G}}$ for the $\widehat{G}$-crystal $\mathbb{B}_{\mu_1} \otimes \cdots \otimes \mathbb{B}_{\mu_d}$. Note that this is equal to $\mathbb{B}_{\mu_\bullet}^{\widehat{G}^d}$ as a set. As a $\widehat{G}$-crystal, we can decompose $\mathbb{B}_{\mu_\bullet}^{\widehat{G}}$ into simple objects, i.e., $\mathbb{B}_{\mu_\bullet}^{\widehat{G}} = \sqcup_\mu \mathbb{B}_\mu^{m_{\mu_\bullet}^\mu}$. Here $m_{\mu_\bullet}^\mu$ denotes the multiplicity with which $\mathbb{B}_\mu$ appears in $\mathbb{B}_{\mu_\bullet}^{\widehat{G}}$. Using this decomposition, we define a natural map

$$\otimes\colon \mathbb{B}_{\mu_\bullet}^{\widehat{G}^d} \to \mathbb{B}_{\mu_\bullet}^{\widehat{G}} \to \sqcup_\mu \mathbb{B}_\mu$$

as a composition of the map given by taking tensor product and the canonical projection to highest weight $\widehat{G}$-crystals.

For $1 \leq k < n$, let ω_k be the cocharacter of the form $(1, \ldots, 1, 0, \ldots, 0)$ in which 1 is repeated k times. Assume that each μ_i is equal to ω_{k_i} for some $1 \leq k_i < n$ and $i < j$ if and only if $k_i \leq k_j$. In the rest of article, we call such $\mu_\bullet$ *Far-Eastern*. Since $\mu_\bullet$ is Far-Eastern, then $|\mu_\bullet| := \mu_1 + \cdots + \mu_d$ is dominant and its last entry is 0. Set $\mu = |\mu_\bullet|$ for some Far-Eastern $\mu_\bullet$. Using Theorem 3.2, we obtain an embedding

$$\mathrm{FE}\colon \mathbb{B}_\mu \to \mathbb{B}_{\mu_\bullet}^{\widehat{G}},$$

which decomposes $\mathbf{b} \in \mathbb{B}_\mu$ into the tensor product of its columns from right to left. By forgetting the $\widehat{G}$-crystal structure, we obtain a map $\mathbb{B}_\mu \to \mathbb{B}_{\mu_\bullet}^{\widehat{G}^d}$, which is also denoted by FE. One can easily verify the following lemma.

Lemma 3.4. For any $\mathbf{b} \in \mathbb{B}_\mu$, $\mathrm{FE}(\mathbf{b})$ is the unique element in $\mathbb{B}_{\mu_\bullet}^{\widehat{G}^d}$ such that $\otimes(\mathrm{FE}(\mathbf{b})) = \mathbf{b}$.

4 Semi-Modules and Crystal Bases

Keep the notations and assumptions above.

4.1 Irreducible Components

Let $\lambda \in X_*(T)$ and $\alpha \in \Phi$. We set $\lambda_\alpha = \langle \alpha, \lambda \rangle$ if $\alpha \in \Phi_-$ and $\lambda_\alpha = \langle \alpha, \lambda \rangle - 1$ if $\alpha \in \Phi_+$. Let U_λ be the subgroup of G generated by U_α such that $\lambda_\alpha \geq 0$. We define $\upsilon_\lambda \in W_0$ to be the unique element such that $U_\lambda = \upsilon_\lambda U \upsilon_\lambda^{-1}$. Here U denotes the unipotent radical of B. It is easy to check $\upsilon_{\eta\lambda} = \tau \upsilon_\lambda$. For $\lambda_\bullet = (\lambda_1, \dots, \lambda_d) \in X_*(T)^d$, set $\upsilon_{\lambda_\bullet} = (\upsilon_{\lambda_1}, \dots, \upsilon_{\lambda_d})$.

Let us denote by $\mathrm{Irr}\, X_{\mu_\bullet}(b_\bullet)$ the set of irreducible components of $X_{\mu_\bullet}(b_\bullet)$. Through the identification $J_b(F) \cong J_{b_\bullet}(F)$ given by $g \mapsto (g, \dots, g)$, this set is equipped with an action of $J_b(F)$. Set $J_b(F)^0 = J_b(F) \cap K = J_b(F) \cap I$. Then we have $J_b(F)/J_b(F)^0 = \{\eta^k J_b(F)^0 \mid k \in \mathbb{Z}\}$ (cf. [1, Lemma 3.3]).

We first consider the case where $\mu_\bullet$ is minuscule. For $\lambda_\bullet \in X_*(T)^d$, set $\lambda_\bullet^\dagger = b_\bullet \sigma_\bullet(\lambda_\bullet)$, $\lambda_\bullet^\natural = \lambda_\bullet^\dagger - \lambda_\bullet$ and $\lambda_\bullet^\flat = \upsilon_{\lambda_\bullet}^{-1}(\lambda_\bullet^\natural)$. It is easy to check $(\eta\lambda_\bullet)^\flat = \lambda_\bullet^\flat$. Let λ_b denote the cocharacter whose i-th entry is $\lfloor \frac{im}{n} \rfloor - \lfloor \frac{(i-1)m}{n} \rfloor$.

Theorem 4.1. Assume that $\mu_\bullet \in X_*(T)_+^d$ is minuscule. Then $\lambda_\bullet \in \mathcal{A}_{\mu_\bullet, b_\bullet}^{\mathrm{top}}$ if and only if $\lambda_\bullet^\flat \in \mathbb{B}_{\mu_\bullet}^{\widehat{G}^d}(\lambda_b)$, and $X_{\mu_\bullet}^{\lambda_\bullet}(b_\bullet)$ is an affine space for such $\lambda_\bullet$. Moreover, the maps $\lambda_\bullet \mapsto \lambda_\bullet^\flat$ and $\lambda_\bullet \mapsto \overline{X_{\mu_\bullet}^{\lambda_\bullet}(b_\bullet)}$ induce bijections

$$J_b(F) \backslash \mathrm{Irr}\, X_{\mu_\bullet}(b_\bullet) \cong \mathbb{A}_{\mu_\bullet, b_\bullet}^{\mathrm{top}} \cong \mathbb{B}_{\mu_\bullet}^{\widehat{G}^d}(\lambda_b).$$

Proof. This follows from [7, Proposition 2.9 & Theorem 3.3]. Note that we have $\mathrm{Stab}_{J_b(F)}(X_{\mu_\bullet}^{\lambda_\bullet}(b_\bullet)) = J_b(F)^0$. □

We write $\gamma^{G^d} \colon \mathrm{Irr}\, X_{\mu_\bullet}(b_\bullet) \to \mathbb{B}_{\mu_\bullet}^{\widehat{G}^d}$ for the map which factors through this bijection. Set $\mu = |\mu_\bullet|$. By [7, Corollary 1.6], the projection $\mathrm{pr} \colon \mathcal{G}r^d \to \mathcal{G}r$ to the first factor induces a $J_b(F)$-equivariant map

$$\mathrm{Irr}\, X_{\mu_\bullet}(b_\bullet) \to \sqcup_{\mu' \leq \mu} \mathrm{Irr}\, X_{\mu'}(b), \quad C \mapsto \mathrm{pr}(C),$$

which is also denoted by pr. The general case can be characterized by the minuscule case using pr and the tensor product of $\widehat{G}$-crystals:

Theorem 4.2. There exists a map

$$\gamma^G \colon \mathrm{Irr}\, X_\mu(b) \to \mathbb{B}_\mu(\lambda_b)$$

which is characterized by the Cartesian square

$$\begin{array}{ccc} \mathrm{Irr}\, X_{\mu_\bullet}(b_\bullet) & \xrightarrow{\gamma^{G^d}} & \mathbb{B}_{\mu_\bullet}^{\widehat{G}^d} \\ {\scriptstyle \mathrm{pr}} \downarrow & & \downarrow {\scriptstyle \otimes} \\ \sqcup_{\mu' \leq \mu} \mathrm{Irr}\, X_{\mu'}(b) & \xrightarrow{\gamma^G} & \sqcup_{\mu' \leq \mu} \mathbb{B}_{\mu'}^{\widehat{G}}, \end{array}$$

where $\mu_\bullet$ is a minuscule cocharacter in $X_*(T)_+^d$ such that $\mu = |\mu_\bullet|$. Moreover, γ^G factors through a bijection

$$J_b(F)\backslash \operatorname{Irr} X_\mu(b) \cong \mathbb{B}_\mu(\lambda_b).$$

Proof. This follows from [7, Theorem 0.5 & Theorem 0.7]. □

Let us denote by Γ^{G^d} (resp. Γ^G) the bijection $\mathbb{A}^{\mathrm{top}}_{\mu_\bullet,b_\bullet} \to \mathbb{B}^{\widehat{G}^d}_{\mu_\bullet}(\lambda_b)$ (resp. $\mathbb{A}^{\mathrm{top}}_{\mu,b} \to \mathbb{B}_\mu(\lambda_b)$) induced by γ^{G^d} (resp. γ^G). Then by Theorem 4.1 and Theorem 4.2, we have the Cartesian square

$$\begin{array}{ccc} \mathbb{A}^{\mathrm{top}}_{\mu_\bullet,b_\bullet} & \xrightarrow{\Gamma^{G^d}} & \mathbb{B}^{\widehat{G}^d}_{\mu_\bullet}(\lambda_b) \\ \Big\downarrow{\scriptstyle \mathrm{pr}} & & \Big\downarrow{\scriptstyle \otimes} \\ \bigsqcup_{\mu' \le \mu} \mathbb{A}^{\mathrm{top}}_{\mu',b} & \xrightarrow{\Gamma^{G}} & \bigsqcup_{\mu' \le \mu} \mathbb{B}^{\widehat{G}}_{\mu'}(\lambda_b), \end{array}$$

where $\mu_\bullet$ is a minuscule cocharacter in $X_*(T)_+^d$ such that $\mu = |\mu_\bullet|$.

4.2 Construction

Let $\mu \in X_*(T)_+$. For $1 \le k \le \mu(1)$, set

$$\mu_k = \begin{cases} \omega_1 & (1 \le k \le \mu(1) - \mu(2)), \\ \omega_2 & (\mu(1) - \mu(2) < k \le \mu(1) - \mu(3)), \\ \vdots & \\ \omega_{n-2} & (\mu(1) - \mu(n-2) < k \le \mu(1) - \mu(n-1)), \\ \omega_{n-1} & (\mu(1) - \mu(n-1) < k \le \mu(1)). \end{cases}$$

Set $d = \mu(1)$. Obviously $\mu_\bullet \in X_*(T)_+^d$ is Far-Eastern (§3.2) and $\mu = |\mu_\bullet|$.

Let $w_{\max}$ denote the maximal length element in W_0. Set $\lambda_b^{\mathrm{op}} = w_{\max}\lambda_b$. For any $\mathbf{b} \in \mathbb{B}_\mu(\lambda_b)$, we denote by $\mathbf{b}^{\mathrm{op}}$ the conjugate of $\mathbf{b}$ with weight λ_b^{op}. Let $1 \le m_0 < n$ be the residue of m modulo n. Note that each entry of λ_b is $\lfloor \frac{m}{n} \rfloor$ or $\lfloor \frac{m}{n} \rfloor + 1$, and $\lambda_b(i) = \lambda_b(n+1-i)$ for any $2 \le i \le n-1$. Let $i_0 = 1 < i_1 < i_2 < \cdots < i_{m_0} = n$ be the integers such that $\lambda_b(i_1) = \lambda_b(i_2) = \cdots = \lambda_b(i_{m_0}) = \lfloor \frac{m}{n} \rfloor + 1$. Then

$$\lambda_b^{\mathrm{op}} = w'_{\max}\lambda_b, \quad \text{where } w'_{\max} = (s_{i_{m_0-1}} \cdots s_{n-1}) \cdots (s_{i_1} \cdots s_{i_2-1})(s_1 \cdots s_{i_1-1}).$$

Here $\lambda_b(i) = \lfloor \frac{m}{n} \rfloor$ (resp. $\lambda_b(i+1) = \lfloor \frac{m}{n} \rfloor$) if and only if $s_{i-1}s_i \le w'_{\max}$ (resp. $s_i s_{i+1} \le w'_{\max}$). By Lemma 3.3, it follows that $\mathbf{b}^{\mathrm{op}}$ can be computed by the action of the Coxeter element $w'_{\max}$. In this computation, each s_i acts as the action of $\tilde{e}_i$ because $\lfloor \frac{m}{n} \rfloor - (\lfloor \frac{m}{n} \rfloor + 1) = -1$. Therefore, if we write

$$\mathrm{FE}(\mathbf{b}) = \mathbf{b}_1 \otimes \cdots \otimes \mathbf{b}_d,$$

then there exists $(w_1, \ldots, w_d) \in W_0^d$ such that

$$\mathrm{FE}(\mathbf{b}^{\mathrm{op}}) = w_1\mathbf{b}_1 \otimes \cdots \otimes w_d\mathbf{b}_d$$

and each simple reflection appears exactly once in some $\mathrm{supp}(w_j)$. One can easily verify the following lemma.

Lemma 4.3. The tuple $(w_1, \ldots, w_d) \in W_0^d$ as above is uniquely determined by $\mathbf{b}$. In particular, $w(\mathbf{b}) := w_1^{-1} \cdots w_d^{-1}$ is a Coxeter element uniquely determined by $\mathbf{b}$.

We call $w(\mathbf{b})$ the *Coxeter element associated to* $\mathbf{b}$. Set $\Upsilon(\mathbf{b}) = \{\upsilon \in W_0 \mid \upsilon^{-1}\tau^m\upsilon = w(\mathbf{b})\}$, where $\tau = s_1 s_2 \cdots s_{n-1}$. Clearly $|\Upsilon(\mathbf{b})| = n$.

For any $\mathbf{b}' \in \mathbb{B}_\mu$, set

$$\xi(\mathbf{b}') = (\varepsilon_1(\mathbf{b}') + \cdots + \varepsilon_{n-1}(\mathbf{b}'), \varepsilon_2(\mathbf{b}') + \cdots + \varepsilon_{n-1}(\mathbf{b}'), \ldots, \varepsilon_{n-1}(\mathbf{b}'), 0).$$

Let λ_b^- be the anti-dominant conjugate of λ_b, and let $\mathbf{b}^-$ be the conjugate of $\mathbf{b}$ with weight λ_b^-. For any $\mathbf{b} \in \mathbb{B}_\mu(\lambda_b)$ and $\upsilon \in \Upsilon(\mathbf{b})$, we define $\xi_\bullet(\mathbf{b}, \upsilon) \in X_*(T)^d$ by

$$\xi_j(\mathbf{b}, \upsilon) = \upsilon\xi(\upsilon^{-1}\mathbf{b}^-) + \sum_{1 \le j' < j} \upsilon w_1^{-1} \cdots w_{j'-1}^{-1}\, \mathrm{wt}(\mathbf{b}_{j'}) \quad (1 \le j \le d).$$

Theorem 4.4. We have $\upsilon_{\xi_j(\mathbf{b},\upsilon)} = \upsilon w_1^{-1} \cdots w_{j-1}^{-1}$ and $\xi_\bullet(\mathbf{b}, \upsilon) \in \mathcal{A}^{\mathrm{top}}_{\mu_\bullet, b_\bullet}$. Moreover, if υ' is an element in $\Upsilon(\mathbf{b})$ different from υ', then $\xi_\bullet(\mathbf{b}, \upsilon) \neq \xi_\bullet(\mathbf{b}, \upsilon')$ and $\xi_\bullet(\mathbf{b}, \upsilon) \sim \xi_\bullet(\mathbf{b}, \upsilon')$. Finally, we have

$$(\Gamma^{G^d})^{-1}(\mathrm{FE}(\mathbf{b})) = [\xi_\bullet(\mathbf{b}, \upsilon)].$$

Remark 4.5. Clearly, this construction itself does not depend on the choice of realization of $\mathbb{B}_\mu$.

We can prove Theorem 4.4 purely combinatorially, using Young tableaux. See [9] for details.

4.3 An Example

In this subsection, we give an example. We consider the case for $n = 5, m = 12$ and $\mu = (4, 3, 3, 2, 0)$. Then $\mu_1 = (1, 0, 0, 0, 0), \mu_2 = (1, 1, 1, 0, 0), \mu_3 = (1, 1, 1, 1, 0), \mu_4 = (1, 1, 1, 1, 0), \lambda_b = (2, 2, 3, 2, 3)$ and $\lambda_b^{\mathrm{op}} = (3, 2, 3, 2, 2)$. Set

$$\mathbf{b} = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 3 & 3 \\ \hline 2 & 2 & 4 \\ \cline{1-3} 3 & 4 & 5 \\ \cline{1-3} 5 & 5 \\ \cline{1-2} \end{array} \in \mathbb{B}_\mu(\lambda_b).$$

Then

$$\mathrm{FE}(\mathbf{b}) = \mathbf{b}_1 \otimes \mathbf{b}_2 \otimes \mathbf{b}_3 \otimes \mathbf{b}_4 = \begin{array}{|c|}\hline 3 \\ \hline\end{array} \otimes \begin{array}{|c|}\hline 3 \\ \hline 4 \\ \hline 5 \\ \hline\end{array} \otimes \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 4 \\ \hline 5 \\ \hline\end{array} \otimes \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline 5 \\ \hline\end{array} \in \mathbb{B}_{\mu_\bullet}^{\widehat{G}^d}.$$

By Theorem 4.1, we want to find $\lambda_\bullet$ satisfying

$$\begin{aligned}
&[\lambda_\bullet] = (\Gamma^{G^d})^{-1}(\mathrm{FE}(\mathbf{b})) \\
\Leftrightarrow\ & \lambda_\bullet^\flat = \mathrm{FE}(\mathbf{b}) \in \mathbb{B}_{\mu_\bullet}^{\widehat{G}^d}(\lambda_b) \\
\Leftrightarrow\ & \upsilon_{\lambda_1}^{-1}(\lambda_2 - \lambda_1) = \mathrm{wt}(\mathbf{b}_1) = (0,0,1,0,0), \\
& \upsilon_{\lambda_2}^{-1}(\lambda_3 - \lambda_2) = \mathrm{wt}(\mathbf{b}_2) = (0,0,1,1,1), \\
& \upsilon_{\lambda_3}^{-1}(\lambda_4 - \lambda_3) = \mathrm{wt}(\mathbf{b}_3) = (1,1,0,1,1), \\
& \upsilon_{\lambda_4}^{-1}(b\lambda_1 - \lambda_4) = \mathrm{wt}(\mathbf{b}_4) = (1,1,1,0,1).
\end{aligned}$$

In the sequel, we check that for $\upsilon \in \Upsilon(\mathbf{b})$, $\lambda_\bullet = \xi_\bullet(\mathbf{b}, \upsilon)$ satisfies these equations. Since

$$\mathbf{b}^{\mathrm{op}} = \tilde{e}_3\tilde{e}_4\tilde{e}_1\tilde{e}_2\mathbf{b} = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 3 \\ \hline 2 & 2 & 4 \\ \cline{1-3} 3 & 3 & 5 \\ \cline{1-3} 4 & 5 \\ \cline{1-2}\end{array} \in \mathbb{B}_\mu(\lambda_b^{\mathrm{op}}),$$

we have

$$\mathrm{FE}(\mathbf{b}^{\mathrm{op}}) = \begin{array}{|c|}\hline 3 \\ \hline\end{array} \otimes s_1 s_2 \begin{array}{|c|}\hline 3 \\ \hline 4 \\ \hline 5 \\ \hline\end{array} \otimes s_3 \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 4 \\ \hline 5 \\ \hline\end{array} \otimes s_4 \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline 5 \\ \hline\end{array} \in \mathbb{B}_{\mu_\bullet}^{\widehat{G}^d},$$

and

$$w_1 = 1, w_2 = s_1 s_2, w_3 = s_3, w_4 = s_4, w(\mathbf{b}) = w_1^{-1}w_2^{-1}w_3^{-1}w_4^{-1} = s_2 s_1 s_3 s_4.$$

So

$$\begin{aligned}
\Upsilon(\mathbf{b}) &= \{\upsilon \in W_0 \mid \upsilon^{-1}\tau^{12}\upsilon = s_2 s_1 s_3 s_4\} \\
&= \{\upsilon \in W_0 \mid (1\ 3\ 5\ 2\ 4) = (\upsilon(1)\ \upsilon(3)\ \upsilon(4)\ \upsilon(5)\ \upsilon(2))\} \\
&= \{(1\ 3\ 5\ 4\ 2), (2\ 4\ 5), (1\ 5)(2\ 3), (1\ 2\ 5\ 3\ 4), (1\ 4\ 3)\}.
\end{aligned}$$

Set $\upsilon_1 = (1\ 3\ 5\ 4\ 2), \upsilon_2 = (2\ 4\ 5), \upsilon_3 = (1\ 5)(2\ 3), \upsilon_4 = (1\ 2\ 5\ 3\ 4), \upsilon_5 = (1\ 4\ 3)$. Then

$$\begin{aligned}
&\upsilon_1^{-1}\lambda_b^- = (2,2,3,2,3), \upsilon_2^{-1}\lambda_b^- = (2,3,2,3,2), \upsilon_3^{-1}\lambda_b^- = (3,2,2,3,2), \\
&\upsilon_4^{-1}\lambda_b^- = (2,3,3,2,2), \upsilon_5^{-1}\lambda_b^- = (3,2,2,2,3).
\end{aligned}$$

The corresponding conjugates of $\mathbf{b}$ (cf. [6, Theorem 3.4.2]) are

$$\mathbf{b} = \begin{array}{|c|c|c|c|}\hline 1&1&3&3\\ \hline 2&2&4\\ \cline{1-3} 3&4&5\\ \cline{1-3} 5&5\\ \cline{1-2}\end{array}, \quad \tilde{e}_2\tilde{e}_4\mathbf{b} = \begin{array}{|c|c|c|c|}\hline 1&1&2&3\\ \hline 2&2&4\\ \cline{1-3} 3&4&5\\ \cline{1-3} 4&5\\ \cline{1-2}\end{array}, \quad \tilde{e}_1\tilde{e}_2\tilde{e}_4\mathbf{b} = \begin{array}{|c|c|c|c|}\hline 1&1&1&3\\ \hline 2&2&4\\ \cline{1-3} 3&4&5\\ \cline{1-3} 4&5\\ \cline{1-2}\end{array},$$

$$\tilde{e}_3\tilde{e}_2\tilde{e}_4\mathbf{b} = \begin{array}{|c|c|c|c|}\hline 1&1&2&3\\ \hline 2&2&4\\ \cline{1-3} 3&3&5\\ \cline{1-3} 4&5\\ \cline{1-2}\end{array}, \quad \tilde{e}_1\tilde{e}_2\mathbf{b} = \begin{array}{|c|c|c|c|}\hline 1&1&1&3\\ \hline 2&2&4\\ \cline{1-3} 3&4&5\\ \cline{1-3} 5&5\\ \cline{1-2}\end{array},$$

respectively. From this, we compute

$$\begin{aligned}&\xi(\upsilon_1^{-1}\mathbf{b}^-) = (3,3,1,1,0), \xi(\upsilon_2^{-1}\mathbf{b}^-) = (3,2,1,0,0), \xi(\upsilon_3^{-1}\mathbf{b}^-) = (2,2,1,0,0),\\ &\xi(\upsilon_4^{-1}\mathbf{b}^-) = (3,2,1,1,0), \xi(\upsilon_5^{-1}\mathbf{b}^-) = (3,3,2,1,0),\end{aligned}$$

and

$$\begin{aligned}&\upsilon_1\xi(\upsilon_1^{-1}\mathbf{b}^-) = (3,1,3,0,1), \upsilon_2\xi(\upsilon_2^{-1}\mathbf{b}^-) = (3,0,1,2,0), \upsilon_3\xi(\upsilon_3^{-1}\mathbf{b}^-) = (0,1,2,0,2),\\ &\upsilon_4\xi(\upsilon_4^{-1}\mathbf{b}^-) = (1,3,0,1,2), \upsilon_5\xi(\upsilon_5^{-1}\mathbf{b}^-) = (2,3,1,3,0).\end{aligned}$$

Note that

$$\begin{aligned}&\upsilon_2\xi(\upsilon_2^{-1}\mathbf{b}^-) = \eta(\upsilon_3\xi(\upsilon_3^{-1}\mathbf{b}^-)), \upsilon_4\xi(\upsilon_4^{-1}\mathbf{b}^-) = \eta(\upsilon_2\xi(\upsilon_2^{-1}\mathbf{b}^-)),\\ &\upsilon_1\xi(\upsilon_1^{-1}\mathbf{b}^-) = \eta(\upsilon_4\xi(\upsilon_4^{-1}\mathbf{b}^-)), \upsilon_5\xi(\upsilon_5^{-1}\mathbf{b}^-) = \eta(\upsilon_1\xi(\upsilon_1^{-1}\mathbf{b}^-)).\end{aligned}$$

We first consider the case for υ_3. Set $\xi_\bullet = \xi_\bullet(\mathbf{b}, \upsilon_3)$. Then

$$\begin{aligned}&\xi_1 = (0,1,2,0,2),\\ &\xi_2 = \xi_1 + \upsilon_3 \operatorname{wt}(\mathbf{b}_1) = (0,2,2,0,2),\\ &\xi_3 = \xi_2 + \upsilon_3 \operatorname{wt}(\mathbf{b}_2) = (1,3,2,1,2),\\ &\xi_4 = \xi_3 + \upsilon_3 s_2 s_1 \operatorname{wt}(\mathbf{b}_3) = (2,4,2,2,3).\end{aligned}$$

We can check that

$$\upsilon_{\xi_1} = \upsilon_3, \upsilon_{\xi_2} = \upsilon_3 = \upsilon_3 w_1^{-1}, \upsilon_{\xi_3} = \upsilon_3 s_2 s_1 = \upsilon_3 w_1^{-1} w_2^{-1}, \upsilon_{\xi_4} = \upsilon_3 s_2 s_1 s_3 = \upsilon_3 w_1^{-1} w_2^{-1} w_3^{-1},$$

and

$$\begin{aligned} b\xi_1 - \xi_4 &= \tau^{12}\xi_1 + (3,3,2,2,2) - \xi_4\\ &= (0,2,0,1,2) + (3,3,2,2,2) - (2,4,2,2,3)\\ &= (1,1,0,1,1) = \upsilon_{\xi_4} \operatorname{wt}(\mathbf{b}_4).\end{aligned}$$

Thus $\xi_\bullet^\flat = \mathrm{FE}(\mathbf{b})$. The same holds for other $\upsilon \in \Upsilon(\mathbf{b})$ because $\upsilon_{\eta\lambda} = \tau\upsilon_\lambda$.

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